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NUMERICAL SOLUTION OF THE EQUATIONS FOR A CONTINUOUS MEDIUM REACTOR

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PREFACE

Undertaking a thesis of this nature would have been impossible without the facilities of the Rich Electronic Computer Center. The author wishes to thank all those who have made these facilities available for scientific research.

The author gratefully acknowledges the guidance and inspiration given by Dr. John A. Nohel during the course of this study.

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SUMMARY

This study is concerned with the numerical solution of the nonlinear system of functional-differential equations of the form:

$$\frac{d}{dt} \ln P(t) = - \int_{-\infty}^{\infty} \alpha(x)\, T(x,t)\, dx$$

$$\varepsilon \frac{T(x,t)}{t} = X \frac{\partial^2 T(x,t)}{\partial x^2} + n(x) \,[\, P(t) - 1 \,] \quad ,$$

where P(t) and T(x,t) are the unknown functions, ε and X are given nonnegative constants, $\alpha(x)$ and n(x) are given functions of position, and t is the time. This system is to be solved subject to the initial conditions

$$P(0) = P_0$$

$$T(x,0) = T_0(x) \;, \; -\infty < x < +\infty \;.$$

The problem represented by this system arises in the analysis of dynamic stability of certain types of nuclear reactors. A completely rigorous analytic treatment of the stability problem for reactors of this type is not available, and thus the purpose of this study is twofold: one, to

establish a computationally stable procedure for solving systems of this type and two, using this procedure to verify the dynamic stability of a reactor governed by this system under suitable conditions.

Using an explicit method of solution for the diffusion type equation (involving $T(x,t)$) and a method of numerical integration for the other, a step-by-step procedure for solution of the system is given. The computational stability of each procedure separately and of the union of the two procedures is established.

Finally, two numerical examples are given which illustrate the procedure and provide examples of dynamically stable reactors governed by the given system.

CHAPTER I

INTRODUCTION

This study is concerned with the numerical solution of the nonlinear system of functional-differential equations of the form:

$$\frac{d}{dt} \ln P(t) = - \int_{-\infty}^{\infty} \alpha(x) T(x,t)\, dx$$

$$\varepsilon \frac{T(x,t)}{t} = X \frac{\partial^2 T(x,t)}{\partial x^2} + n(x)[P(t) - 1] \quad , \qquad (1)$$

where $P(t)$ and $T(x,t)$ are the unknown functions; ε and X are given nonnegative constants; $\alpha(x)$ and $n(x)$ are given functions of position; and t is the time. The system is to be solved subject to the initial conditions:

$$P(o) = P_o$$

$$T(x,o) = T_o(x) \quad . \qquad (2)$$

The problem represented by the relations (1) and (2) above arises in the analysis of dynamic stability of certain types of nuclear reactors. The problem is in fact a generalization of a class of reactor problems considered by Ergen, Lipkin and Nohel [1] for which rigorous analytic stability analysis is possible. In that class of problems it is not necessary to resort to numerical procedures. The methods explored in [1] have been modified by Ergen and Nohel [2] so as to apply to the problem considered in this study, but it is not possible to justify this analytical treatment completely rigorously. The result of this study provides a check on the work done by Ergen and Nohel [2] in addition to presenting a method of solution for a new class of problems in numerical analysis typified by the system (1).

The problem of reactor stability arises in the study of reactor control. When a reactor becomes supercritical it must be brought under control rapidly. This cannot be done with control rods alone, and therefore a reactor must be inherently stable--that is, it must have dynamic as well as static stability.

Static stability implies that reactivity decreases with increasing temperature, and is assured by designing the reactor to have a negative temperature coefficient of reactivity. However, since heating in the reactor is a gradual process, the negative temperature coefficient cannot compensate instantaneously for an excess in reactivity. Thus, since the power of the reactor cannot immediately return to normal, there may be oscillations in reactor power. A reactor is said to be dynamically stable if these oscillations do not build up.

The problem of computational stability arises in the study of error propagation in iterative numerical techniques (i.e. techniques in which the result of one step is used in computing the next step) such as are used in the inversion of matrices, or in the step-by-step solution of an initial-value problem in differential equations. The problem of this thesis falls into the latter category. Computational stability will be defined more precisely in Chapter II for the problem of this study. However, basically the concept can be described in the following way. Let S_j be the exact result at step j of a numerical procedure P. Let S_j^* be an approximate result. If the error $S_j - S_j^*$ committed at step j does not grow in magnitude as P is repeated for the whole range of the problem, the procedure is said to be computationally stable; otherwise the procedure P is said to be computationally unstable. Hereafter, unless otherwise stated, stability will mean computational stability.

There is a second problem connected with the solution of partial differential equations. That is the problem of convergence. Let D be the solution of the partial <u>differential</u> equation, and Δ be the solution of the corresponding partial <u>difference</u> equation obtained by approximating the derivatives of the original equation by difference quotients. Then $(D - \Delta)$ is called the truncation error and the problem of convergence is that of finding conditions under which

$(D - \Delta)$ tends to zero as the mesh size tends to zero. This problem had been widely treated, and a classic paper on it is that of Courant, Friedrichs, and Lewy [3]. While convergence is touched on briefly in Chapter II, it is not the main concern of this study. The interested reader is referred to [3] as well as to a discussion in Chapter III of <u>Methods</u> of <u>Applied</u> <u>Mathematics</u> by F. B. Hildebrand [4].

The numerical methods of solution of partial differential equations fall into two general categories: those in which the value to be determined at the n^{th} step are expressed explicitly in terms of known values, and those in which two or more values to be determined at the n^{th} step are related to known values. The former are known as <u>explicit</u> or "step-by-step" methods; the latter are known as <u>implicit</u> methods. The implicit methods require the solution of n-2 simultaneous linear equations--n being the number of mesh points in one direction. Although the implicit methods generally have less truncation error and better stability properties and are the ones most widely used at the present time, they were considered to be impractical for the present problem since the systems of equations involved would have been of too large an order for the particular computing machine available.

It will be observed that the second equation of the system (1) is an inhomogeneous parabolic equation of the heat conduction type. The most general and complete treatment of the solution of parabolic equations by explicit methods was found in a recent paper by Fritz John [5]. In Chapter II the material from [5] is modified so as to be pertinent to the present problem, and a general stability criterion for the solution of the heat equation is presented.

The problem considered in this study, however, requires not only the stability of the procedure to solve the heat equation but also the stability of the overall procedure which solves the system (1). This problem is considered in Chapter III, together with a discussion of the truncation errors of the various numerical approximations used.

Two numerical examples will be considered. The first is a special case of (1) in which $X = 0$, and $n(x)$ and $\alpha(x)$ are constants. This case represents a situation in which $T(x,t)$ is independent of X and is useful in studying the effect of the truncation terms which depend on t. It will also be shown that the solution in this case is periodic; and thus it may be used as a check on the numerical procedure. The second example is one for which $X = 0$. Both problems provide special cases of a continuous medium reactor which has the property of dynamic stability.

CHAPTER II

NUMERICAL SOLUTION OF A PARABOLIC PARTIAL DIFFERENTIAL EQUATION

Preliminary considerations.--One purpose of this chapter is to establish a mathematical definition of stability of numerical procedures for solving the equation:

$$\frac{\partial T(x,t)}{\partial t} = a\ (x,t)\ \frac{\partial^2 T(x,t)}{\partial x^2} + d\ (x,t) \tag{3}$$

with the initial condition

$$T(x,o) = T_o(x) \quad ; \tag{4}$$

the region of consideration is:*

$$R: -\infty < x < \infty \quad , \quad o \leq t \leq t_n \quad .$$

It will be assumed that a(x,t) is positive and bounded away from zero in R and that d(x,t) is bounded in R.

With the definition established, necessary and sufficient conditions for stability which are applicable to

---

*Unless specifically noted otherwise, the capital letter R will refer to this particular region for the remainder of this chapter.

the numerical examples of the later chapters will be given. In this respect it should be noted that equation (3) is somewhat more general than the actual numerical examples solved in this study. However, the more general analysis is presented in order that the results will have a wider area of application.

A difference equation which approximates equation (3) is obtained by replacing the derivatives in (3) with difference quotients. One way of accomplishing this is as follows: If $T(x,t)$ is of class C_{n+1} with respect to t--that is, $T(x,t)$ and its first n+1 partial derivatives with respect to t are continuous--then, by Taylor's formula with remainder,

$$T(x,t+k) = T(x,t) + \sum_{j=1}^{n} \frac{k^j}{j!} \frac{\partial^j T(x,t)}{\partial t^j}$$

$$+ \frac{k^{n+1}}{n+1!} \frac{\partial^{n+1} T(x,\ )}{\partial t^{n+1}} , \qquad (5)$$

for some τ : $t < \tau < t+k$. Now, the derivatives with respect to t in (5) can be replaced by equivalent expressions involving derivatives with respect to x by using equation (3)--assuming sufficient smoothness of $T(x,t)$ with respect to x. If, then, these derivatives with respect to x are replaced by difference quotients, an (explicit) difference equation for the value $T(x,t+k)$ results.

For example, taking n = 1 in equation (5) gives the following result:

$$T(x,t+k) = T(x,t) + k\,\frac{\partial T(x,t)}{\partial t} + E_1(x,t) \quad ,$$

where $E_1(x,t)$ is the truncation error term. Thus, if the second derivatives of $T(x,t)$ is bounded in R then $|E_1(x,t)| \leq Mk^2$ for (x,t) in R and M a positive constant. Substitution for $\frac{\partial T(x,t)}{\partial t}$ as given by (3) yields,

$$T(x,t+k) = T(x,t) + ka(x,t)\,\frac{\partial^2 T}{\partial x^2} + kd(x,t) + E_1(x,t) \quad . \qquad (6)$$

In central difference notation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{h^2}\left[\delta_x^{\,2}T(x,t) - \frac{1}{12}\,\delta_x^{\,4}T(x,t) + \frac{1}{90}\,\delta_x^{\,6}T(x,t) - \ldots\ldots\right],$$

where δ_x is the central difference operator with respect to x (Cf. [6]). Let

$$\frac{\partial^2 T}{\partial x^2} = \frac{\delta_x^{\,2}T(x,t)}{h^2} + E_2(x,t) \quad ,$$

where $E_2(x,t)$ is the truncation error of the difference representation, and substitute in (6). Then

$$T(x,t+k) = \sum_{n=1}^{1} C_r(x,t,h,k)T(x+rh,t) + kd(x,t) + E(x,t) \ , \ (7)$$

where

$$C_{-1} = \frac{ka(x,t)}{h^2}\ ,\ C_o = 1 - \frac{2ka(x,t)}{h^2}\ ,\ C_1 = \frac{ka(x,t)}{h^2}\ ,$$

and

$$E(x,t) = E_1(x,t) + E_2(x,t) \quad .$$

Now let $u(x,t)$ be the solution of the difference equation:

$$u(x,t+k) = \sum_{n=-1}^{1} C_r(x,t,h,k)\ u(x+rh,t) + kd(x,t) \quad , \tag{8}$$

where the coefficient $C_r(x,t,h,k)$ are the same as in (7) and the solution $u(x,t)$ of (8) holds at every point (x,t) of R which is on the "h by k" mesh (or lattice) covering R. Also, let $u(x,t)$ satisfy the initial condition (4). The quantity $E(x,t)$ is the truncation error of the procedure. It will be treated in detail in the later chapters where numerical examples are treated. By taking higher order terms in the expansion (5), and by taking higher order difference approximation of the derivatives one can obtain "better" approximations to equation (3)--that is, approximations in which the truncation error is smaller than the one resulting from the approximation (8). In all cases equation (8) takes the form:

$$u(x,t+k) = \sum_{r=-N}^{N} C_r(x,t,h)\ u(x+rh,t) + kw(x,t,k) \quad . \tag{9}$$

It will be assumed that $w(x,t,k)$ is bounded in R and that

$$\lim_{k \to 0} w(x,t,k) = d(x,t) \quad .$$

In order to consider the C_r as functions of x,t, and h it is assumed that k is some definite function of h, say g(h), where

$$\lim_{h \to 0} g(h) = 0 \quad .$$

A basic set of conditions on the $C_r(x,t,h)$ under which the solution $u(x,t)$ of (9) approximates the solution $T(x,t)$ of (3) are obtained by writing (9) in the form:*

$$\frac{u(x,t+k) - u(x,t)}{k} = -\frac{1}{k}\, u(x,t) + \frac{1}{k} \sum_r C_r(x,t,h)\, u(x,rh,t)$$

$$+ w(x,t,k) \quad , \tag{10}$$

and requiring that (10) become (3) in the limit as h (and k) approach zero. The conditions thus obtained are necessary conditions for convergence, and they are called "compatability relations" by Fritz John [7], whose development will be followed here. Equation (10) can be written in the form

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*Unless explicitly noted otherwise, all summations with respect to r will extend from -N to +N for the remainder of the thesis.

$$\frac{u(x,t+k) - u(x,t)}{k} = \frac{1}{k} \sum_r (C_r(x,t,h) - \delta_{0r})\, u(x+rh,t)$$

$$+ w(x,t,k) \quad , \tag{11}$$

where δ_{0r} is the Kroenecker delta. If $u(x,t)$ is of class C^2 with respect to x for (x,t) in R, one may write

$$u(x+rh,t) = u(x,t) + rh\frac{\partial u(x,t)}{\partial x} + \frac{r^2h^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{rh^2}{2}\, n(x,t,h),$$

where

$$\lim_{h \to 0} n(x,t,h) = 0 \quad .$$

Whence, on substitution in (11),

$$\frac{u(x,t+k) - u(x,t)}{k} = \frac{1}{k} \sum_r (C_r(x,t,h) - \delta_{0r})\, u(x,t)$$

$$+ \frac{1}{k} \sum_r (C_r(x,t,h) - \delta_{0r})\, rh\frac{\partial u(x,t)}{\partial x}$$

$$+ \frac{1}{k} \sum_r (C_r(x,t,h) - \delta_{0r})\, \frac{r^2h^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}$$

$$+ \frac{1}{k} \sum_r (C_r(x,t,h) - \delta_{0r})\, \frac{r^2h^2}{2}\, n(x,t,h)$$

$$+ w(x,t,k) \quad . \tag{12}$$

The requirement that (12) goes over into (3) as h (and k) approach zero implies:

$$\lim_{h\to 0} \frac{1}{k} \sum_r (C_r - \delta_{0r}) = 0 \quad ,$$

$$\lim_{h\to 0} \frac{h}{k} \sum_r r(C_r - \delta_{0r}) = 0 \quad , \tag{13}$$

$$\lim_{h\to 0} \frac{h^2}{k} \sum_r \frac{r^2}{2} (C_r - \delta_{0r}) = a(x,t) \quad .$$

It will now be assumed that the $C_r(x,t,h)$ are bounded in R. Note that this is the case for equation (7) if $a(x,t)$ is bounded in R. Recall that $a(x,t)$ was assumed positive and bounded away from zero in R. Then from the last relation in (13) one obtains the result that $\lim_{h\to 0} \frac{h^2}{k}$ exists and is positive, say $\frac{1}{\lambda}$, and that

$$\sum_r r^2(C_r - \delta_{0r,}) = 2\lambda a(x,t) \quad .$$

In order that

$$\lim_{h\to 0} \frac{k}{h^2} = \lambda \quad , \qquad \lambda > 0 \quad ,$$

exists it is sufficient to assume that $\frac{k}{h^2} = \lambda$. For the purpose of this study it will also be sufficient to assume that once λ , a positive constant, is chosen the $C_r(x,t,h)$ are in fact independent of h and depend only on x, t, and the number λ .

The above considerations lead to the following result:

Theorem 1. The difference equation (9) will be compatible with the differential equation (3) if:

$$k/h^2 = \lambda \ , \qquad \lambda > 0 \quad ,$$

$C_r(x,t)$ are uniformly bounded in R,

$$\sum_r r^2(C_r(x,t) = 2\lambda a(x,t) \tag{14}$$

$$\sum_r r(C_r(x,t) = 0$$

$$\sum_r C_r(x,t) = 1 \quad .$$

Stability considerations.--For a given function g(x) and for every m and h let the operator L_{mh} be defined by the following relations:

$$L_{mh}[\ g(x)\ ] \equiv 0 \qquad \text{for } t < (m+1)k \quad , \tag{15}$$

$$L_{mh}[\ g(x)\ ] \equiv g(x) \qquad \text{for } t = (m+1)k \quad , \tag{16}$$

$$L_{mh}[\ g(x)\ ] \equiv v(x,t) \qquad \text{for } t \geq (m+1)k \quad ,$$

where v(x,t) satisfied the homogeneous difference equation

$$v(v,t+k) - \sum_r C_r(x,t)\ v(x+rh,t) = 0 \tag{17}$$

The solution of the inhomogeneous equation (9) can then be expressed by means of superposition of solutions of the form (15) and (16) which employs the operator L_{mh}. Consider the inhomogeneous equation:

$$u(x,t+k) = \sum_r C_r(x,t)\; u(x+rh,t) + kw(x,t,k) \quad , \tag{18}$$

where

$$u(x,0) = u_0(x) \quad .$$

Take the sequence

$$L_{1h} \;[\; u_0(x) \;] \quad ,$$

$$L_{0h} \;[\; w(x,0,k \;] \quad ,$$

$$\vdots$$

$$L_{mh} \;[\; w(x,t_m,k) \;] \quad ,$$

$$\vdots$$

and consider the solution u(x,t) of (18) at any given time, say t = Mk. Then from (15)

$$L_{mh} \;[\; w(x,t_m,k) \;] \equiv 0 \qquad \text{for } m \geq M \quad .$$

<u>Theorem 2</u>. The solution u(x,t) of equation (18) is

given by:

$$u(x,t) = \sum_{m=-1}^{M-1} L_{mh} [f_m(x)] \quad , \tag{19}$$

where

$$f_m(x) = \begin{cases} u_0(x) & m = -1 \\ kw(x,t_m,k) & m \geq 0 \end{cases} \quad . \tag{20}$$

<u>Proof</u>: The proof follows easily from direct substitution of (19) in (18). Now,

$$u(x,t+k) = \sum_{m=-1}^{M} L_{mh} [f_m(x)] \quad ,$$

and

$$u(x,t) = \sum_{m=-1}^{M-1} L_{mh} [f_m(x)] \quad ,$$

for t = Mk. Thus

$$\sum_{m=-1}^{M-1} L_{mh} [f_m(x)] + L_{mh} [f_m(x)]$$

$$= \sum_{r} C_r(x,t) \sum_{m=-1}^{M-1} L_{mh} [f_m(x)] + kw(x,t,k) \quad .$$

Since all sums of this equation are finite it may be rearranged as follows:

$$\sum_{m=-1}^{M-1} \left[L_{mh} [f_m(x)] - \sum_r C_r(x,t) L_{mh} [f_m(x)] \right]$$

$$+ L_{mh} [f_m(x)] = kw(x,t,k) \tag{21}$$

Now, by the third property of L_{mh} [g] as given by (16) the quantity in brackets is zero for m=-1, 0, ..., M-1; moreover, from (20),

$$L_{mh} [f_m (x)] = kw(x,t,k)$$

for t = Mk. Hence (21) reduces to the identity:

$$kw(x,t,k) = kw(x,t,k) \quad ,$$

and thus u(x,t) as given by (19) is a solution of (18).
Q. E. D.

The <u>norm</u> of the operator, $L_{mh} = v(x,t)$, is defined as,

$$\| L_{mh} \| = \| v(x,t) \| = \underset{R_h}{\text{l.u.b.}} \left| v(x,t) \right| \tag{22}$$

where R_h denotes the totality of mesh points in the region R. The <u>norm</u> of a function g(x) is defined as:

$$\| g(x) \| = \underset{s}{\text{l.u.b.}} \left| g(sh) \right| \quad , \tag{23}$$

where s extends over all the integers. Equation (18) is defined to be <u>stable</u> if the operators L_{mh} are uniformly bounded (in the sense of the norm (22)) in m and h for

$$\begin{aligned} k &= \lambda h^2 \\ 0 \leq t &\leq t_n \quad . \end{aligned} \tag{24}$$

From the first relation in (16) note that for $t < 0$ it follows that $L_{mh}[g] \equiv 0$ for all m. Thus (24) may be equivalently written

$$\begin{aligned} k &= \lambda h^2 \\ -k \leq t &\leq t_n \quad . \end{aligned} \tag{25}$$

Equivalently, equation (18) is stable if there exists a positive constant Q, independent of m and h, such that for every u(x,t) satisfying the homogeneous equation (17) it is true that

$$\left| v(x,t) \right| \leq Q \ \underset{x}{\text{l.u.b.}} \ \left| v(x,t_0) \right| \tag{26}$$

for $t_0 \leq t \leq t_n$.

Assume equation (18) is stable, and let u(x,t) be written in terms of the operators L_{mh}. Then

$$u(x,t) = L_{-1h}[\ u_0(x)\ ] + k \sum_{m=0}^{M-1} L_{mh}\ [\ w(x,mk)\ ] \quad ,$$

where t = Mk. Applying the definition of stability (26) gives

$$\left|u(x,t)\right| \leq Q \text{ l.u.b. } \left|u_0(x)\right| + kQ \underset{x}{\text{l.u.b.}} \left|w(x,0)\right| + \ldots$$

$$+ kQ \underset{x}{\text{l.u.b.}} \left|w(x,mk)\right| ,$$

or

$$\left|u(x,t)\right| \leq Q(\underset{x}{\text{l.u.b.}} \left|u_0(x)\right| + Mk \underset{R_h}{\text{l.u.b.}} \left|w(x,mk)\right|) .$$

Using the definitions (22) and (23), this may be written

$$|u(x,t)| \leq Q(\left\|u_0(x)\right\| + t \left\|w(x,mk)\right\|) .$$

And since this is true for all t in R it follows that

$$\left\|u(x,t)\right\| \leq Q \left\|u_0\right\| + Qt_n \left\|w\right\| . \tag{27}$$

Thus if equation (18) is stable the solution u(x,t) is bounded in R, and moreover a bound can be given in terms of the bounds of $u_0(x)$ and w(x,t,k).

Recall that in Chapter I stability was defined to mean that errors committed at one step of the computation do not grow without bound as the computation is carried forward. The relations (22), (23), (26), and (27) imply stability of (18) in this sense; for let u(x,t) be the exact solution of (18) at a given step corresponding to t_0--there is no loss of generality in assuming $t_0 = 0$--

and let $v(x,t)$ be an approximate solution, then

$$u(x,t+k) = \sum_r C_r(x,t)\ u(x+rh,t) + kw(x,t) \quad ,$$

$$v(x,t+k) = \sum_r C_r(k,t)\ v(x+rh,t) + kw(x,t) \quad .$$

Let

$$e(x,t) = u(x,t) - v(x,t) \quad ,$$

and then

$$e(x,t+k) = \sum_r C_r(x,t)\ e(x+rh,t) \quad . \tag{28}$$

Thus the error--from any source--at a given step will be propagated by the homogeneous equation, and the total error at time t can be taken as a superposition of such "error" functions. Thus the definition of stability in terms of the boundedness--in the sense of the norm (22), (23)--of the solutions of the homogeneous difference equation assures that an error committed at step j will not grow without bound, and in fact the total error increases at most like the number of steps taken in the t direction. In particular if $E_m(x)$ represents an error distribution at $t = mk$, and $E_m = \underset{x}{\text{l.u.b.}} \left| E_m(x) \right|$; then

$$\left\| e_m(x,t) \right\| \leq Q\, E_m$$

for some $Q \geq 0$, and $m = 0, 1, \ldots, M-1$; where $e_m(x,t)$ is the error in $v(x,t)$ for $t = Mk > mk$, due to an error distribution $E_m(x)$ at $t = mk$. Let $E = \max\limits_{0 \leq m \leq M} E_m$. Let E_M be the least upper bound of the error in $v(x,t)$ at step $t = Mk$. Then

$$E_M \leq QME \tag{29}$$

Before stating necessary and sufficient conditions for stability, consider the following example. In equation (3) let $d(x,t) \equiv 0$ and $a(x,t) \equiv 1$. Corresponding to equation (8) the following difference equation is obtained.

$$u(x,t+k) = \lambda(u\,(x+h,t) + u(x-h,t)\,) + (1-2\lambda\,)\,u(x,t). \tag{30}$$

Assume that $u(x,t)$ satisfies the initial condition:

$$u(x,0) = e^{iwx}\quad , \tag{31}$$

where the domain of x is the entire real axis, and w is an arbitrary real number. The coefficients C_r for (30) are then,

$$C_{-1} = C_1 = \lambda\quad , C_0 + 1 - 2\lambda\quad , C_r \equiv 0\quad |r| > 1\quad .$$

These quantities are easily seen to satisfy the "compatibility relations" (14) corresponding to (8) with $d(x,t) \equiv 0$,

$a(x,t) \equiv 1$. For equation (30) a solution may be readily obtained by a separation of variables method. Let

$$u(x,t) = f(x)\, g(t) \quad .$$

then the initial condition (31) requires that

$$f(x) = e^{iwx} \tag{32}$$

$$g(0) = 1 \tag{33}$$

Equation (30) becomes on substitution for $u(x,t)$:

$$e^{iw(x)} g(t+k) = \lambda(e^{iw(x+h)} + e^{iw(x-h)})\, g(t) + (1-2\lambda)e^{iwx} g(t).$$

then

$$g(t+k) = [\ \lambda(e^{iwh} + e^{-iwh}) + (1-2\lambda)]\ g(t) \quad ,$$

or equivalently

$$g(t+k) = (2\lambda(\cos wh-1) + 1)\ g(t) \quad .$$

Let

$$b = 2\lambda(\cos wh-1) + 1 \quad . \tag{34}$$

Then $g(x)$ satisfies the recursion formula

$$g(t+k) = b\ g(t) \quad ,$$

and thus,

$$g(k) = b\,g(0) = b$$

$$g(2k) = b^2$$

$$\vdots$$

$$g(mk) = b^m$$

$$\vdots \qquad .$$

Then

$$u(x,t) = b^m e^{iwx} = b^{t/k}\ e^{iwx} \quad ; \tag{35}$$

whence

$$\|u(x,t)\| = |b|^m \quad , \text{ since } \quad |e^{iwx}| = 1 \quad .$$

In order that $\|u(x,t)\|$ remain bounded as $m \to \infty$ it is necessary and sufficient that $|b| \leq 1$. Thus it follows that a necessary and sufficient condition for the stability of (30) is that

$$-1 \leq 2\lambda(\cos wh-1) + 1 \leq 1 \quad ,$$

or

$$-2 \leq 2\lambda(\cos wh-1) \leq 0 \quad .$$

The right-hand inequality holds trivially; for the left-hand inequality to hold for <u>all</u> <u>real</u> <u>w</u> it is necessary and sufficient that $\lambda \leq {}^1/2$. This requirement in terms of the coefficients, C_r, of the difference equation (30) is that

$$
\begin{aligned}
&0 < C_{-1} \leq {}^1/2 \\
&0 \leq C_0 < 1 \qquad\qquad (36) \\
&0 < C_1 \leq {}^1/2 \quad .
\end{aligned}
$$

Recall also that the C_r satisfy the compatibility relations (14). For this particular example, then, necessary and sufficient conditions for stability are obtainable in terms of the coefficients of the difference equation. It is to be hoped that such a result is also possible in more general situations, and this in fact is true. The theorems which follow are proved by Fritz John [8], and the interested reader is referred to his paper for the proofs.

Theorem 3. A necessary condition for stability of the difference equation:

$$u(x,t+k,h) = \sum_r C_r(x,t,h)\ u(x+rh,t,h) + kd(x,t) \qquad (37)$$

is that the coefficients $C_r(x,t)$ satisfy the inequality

$$\left| \sum_r C_r(x,t,0)e^{ir\theta} \right| \leq 1 \qquad (38)$$

for all real θ and all (x,t) in R; and provided $C_r(x,t,h)$ are continuous functions of x,t,h.

Note that in this theorem a more general situation is allowed in which the coefficients [may] depend explicitly on h. Recall that previously in this study it was assumed that the coefficients depended only on the mesh ratio λ in addition to x and t. However, the solution of (37) also depends explicitly on h, and hence the notation $u(x,t,h)$. In his paper Fritz John assumes the $C_r(x,t,h)$ to be of the form

$$C_r(x,t,h) = C_{r0}(x,t) + h\, C_{r1}(x,t) + \frac{h^2}{2}\, C_{r2}(x,t,h) \quad , \qquad (39)$$

where

$$\lim_{h \to 0} C_{r2}(x,t,h) = C_{r2}(x,t,0)$$

uniformly for (x,t) in R. Note, then, that the requirement (38) reduces to a requirement on the $C_{r0}(x,t)$ in which again h does not enter explicitly.

Note also that the condition (38) for equation (30) is exactly the condition that the quantity b as given by equation (34) be not greater than one in absolute value. In that example the coefficients are positive constants and this condition for stability is sufficient as well as necessary. However, for the general case a somewhat stronger sufficiency condition is required. This condition is given by the following theorem.

<u>Theorem 4</u>. Let the coefficients $C_r(x,t)$ of the difference equation (37) satisfy the compatibility relations

(14). Assume also that the quantities

$$\left|C_r(x,t)\right| \quad , \quad \left|\frac{\partial C_r(x,t)}{\partial x}\right| \quad , \quad \left|\frac{\partial^2 C_r(x,t)}{\partial x^2}\right| \tag{40}$$

exist and are uniformly bounded in R, and that there exists a positive number M independent of x and t such that

$$\left|\sum_r C_r(x,t)\ e^{ir\theta}\right| \leq e^{-M\theta^2} \quad \text{for} \quad \theta \leq \pi \ . \tag{41}$$

Then the difference equation (37) is stable.

This theorem is stated here for the less general form of the $C_r(x,t)$. It is given in the reference for the form (39), and for that situation the compatibility requirements and smoothness requirements are extended to include the functions $C_{r1}(x,t)$ and $C_{r2}(x,t,h)$. For many applications the C_r will all be positive constants. Theorem 5 which follows shows that in this case equation (37) will be stable. A similar theorem for the more general form (39) of the C_r where the C_{r0} are all positive constants is proved by Fritz John [9].

<u>Theorem 5</u>. The coefficients of the difference equation (27) will satisfy the requirements of Theorem 4 if all the coefficients $C_r(x,t)$ are positive constants, say c_r, and

$$\sum_r c_r = 1 \quad , \tag{42}$$

$$\sum_r rc_r = 0 \quad . \tag{43}$$

Proof: It is clear that the above hypothesis is sufficient to insure that the c_r satisfy the compatibility conditions (14) and also the equations (42) and (43). Now

$$\sum_r c_r e^{ir\theta} \leq \sum_{\substack{r \\ r \neq 0,\,1}} |c_r| \left|e^{ir\theta}\right| + \left|c_0 + c_1 e^{i\theta}\right| . \tag{44}$$

Also,

$$\begin{aligned}
\left|c_0 + c_1 e^{i\theta}\right| &= \sqrt{(c_0 + c_1 \cos\theta)^2 + (c_1 \sin\theta)^2} \quad , \\
&= \sqrt{c_0^2 + 2c_0 c_1 \cos\theta + c_1^2} \quad , \\
&= \sqrt{(c_0 + c_1)^2 + 2c_0 c_1 (\cos\theta - 1} \quad , \\
&= \sqrt{(c_0 + c_1)^2 - 4c_0 c_1 \sin^2 \tfrac{\theta}{2}} \quad ,
\end{aligned} \tag{45}$$

Substitution of (42) and (45) in (44) yields,

$$\left|\sum_r c_r e^{ir\theta}\right| \leq 1 - (c_0 + c_1) + \sqrt{(c_0+c_1)^2 - 4c_0 c_1 \sin^2 \tfrac{\theta}{2}} \quad . \tag{46}$$

Now $(c_0 - c_1)^2 \geq 0$, implies $(c_0+c_1)^2 \geq 4c_0 c_1$, which in turn implies $\dfrac{4c_0 c_1}{(c_0+c_1)^2} \leq 1$. Using the binomial expansion for the square root in (46) gives:

$$\left|\sum_r c_r e^{ir\theta}\right| \leq 1 - (c_0+c_1) + (c_0+c_1)\left(1 - \frac{2c_0c_1}{(c_0+c_1)^2}\sin^2\frac{\theta}{2}\right) .$$

Then

$$\left|\sum_r c_r e^{ir\theta}\right| \leq 1 - \frac{c_0c_1}{c_0+c_1}(1-\cos\theta) \leq 1 + \frac{c_0c_1}{c_0+c_1}\left[-\frac{\theta^2}{2} + \frac{\theta^4}{4}\right],$$

$$\left|\sum_r c_r e^{ir\theta}\right| \leq 1 - \frac{c_0c_1}{c_0+c_1}\left(\frac{1}{2} - \frac{\theta^2}{24}\right)\theta^2 .$$

Now for $Q \leq \pi$, $\frac{\theta^2}{24} < 5/12$; thus

$$\left|\sum_r c_r e^{ir\theta}\right| \leq 1 - \frac{c_0c_1}{c_0+c_1} \frac{\theta^2}{12} \leq e^{-m\theta^2} ,$$

where $0 < M \leq {}^{c_0c_1}/12(c_0+c_1)$. Thus the conditions of Theorem 4 are fulfilled. Q.E.D.

It should be remarked that the foregoing stability considerations apply only to those equations in which the domain of x is the entire real axis. These results do not (necessarily) apply to those equations where the domain of x is a finite interval and modification of the stability criteria is (generally) required. In this study the domain of x is always taken to be the infinite interval unless otherwise stated.

Once stability is established one may, with suitable restrictions, prove not only the convergence of the solution of the difference equation to the solution of the differential equation, but also the existence of a solution to the differential equation. The following theorem on convergence is particularly suited to the purpose of this study.

Theorem 6. Let $T(x,t)$ be a solution of (3) and (4) such that

$$T(x,t) \quad \frac{\partial T}{\partial x} \ , \quad \frac{\partial^2 T}{\partial x^2} \ , \text{ and } \ \frac{\partial T}{\partial t}$$

are uniformly continuous and bounded in R. Let $u(x,t,h)$ be the solution of (7), and assume the stability and compatibility of (37). Then for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ such that

$$\left| u(x,t,h) - T(x,t) \right| \leq \varepsilon$$

for (x,t) in R_h and for all $0 < h \leq \delta$.

The proof of Theorem 6 is given by Fritz John [10], and is omitted here.

It is of interest to note that (provided a sufficient number of bounded derivatives of the C_r , $u_0(x)$, $d(x,t)$ exist) the existence of a solution of (3) and (4) can be proved by using a subsequence of solutions $u(x,t,h)$ corresponding to different values of h tending to zero. Such methods were introduced by Courant, Friedricks, and Lewy [11], and the next theorem is a special case of one proved by Fritz John [12] using this technique.

<u>Theorem 7</u>. Equations (3) and (4) have a solution $T(x,t)$ of class C^4 with respect to x if $a(x,t)$, $T_0(x)$, $d(x,t)$ are of class C^6 with respect to x, $a(x,t)$ is positive and bounded away from zero and $\frac{\partial a(x,t)}{\partial t}$, $\frac{\partial d(x,t)}{\partial t}$ are continuous.

The introduction of the concept of "generalized" solutions allows the relaxing of the conditions of Theorem 7 to a large extent. An extensive treatment is to be found in section six of the paper by Fritz John [13]. For the purposes of this study Theorem 4 and Theorem 5 will suffice and these further considerations are mentioned only for the benefit of the reader interested in more general problems.

CHAPTER III

THE NUMERICAL SOLUTION OF THE SYSTEM

The numerical solution of the power equation.--The considerations thus far have pertained only to the solution of the diffusion type equation in system (1). The concern of this section will be the solution of the first equation of that system, namely

$$\frac{d \ln p(t)}{dt} = - \int_{-\infty}^{\infty} \alpha(x)T(x,t)dx \tag{42}$$

Integrating both sides of this equation with respect to from 0 to t gives,

$$\ln p(t) = \ln p(0) - \int_{0}^{t} \int_{-\infty}^{\infty} \alpha(x)T(x,\tau)dx d\tau \quad . \tag{43}$$

Thus the problem here is simply that of performing a double integration numerically. Methods of numerical integration are in wide use, and the only concern of this section will be a systematic development of the truncation error terms for the methods of double integration which will be used. The reader is assumed to be familiar with the methods of numerical integration for single integrals and with error investigations for these formulas. The two formulas which will be used are the "trapezoidal rule" and "Simpson's rule" formulas:

$$\int_{x_{n-1}}^{x_n} f(x)\,dx = \frac{h}{2}\left[f(x_{n-1}) + f(x_n) \right] - \frac{f''(s)h^3}{12} \tag{44}$$

where $x_{n-1} < s < x_n$, and

$$\int_{x_{n-1}}^{x_{n+1}} f(x)dx = \frac{h}{3}\left[f(x_{n-1}) + 4f(x_n) + f(x_{n+1}) \right] - \frac{f^{(iv)}(x)h^5}{90} \tag{45}$$

where $x_{n-1} < s < x_{n+1}$. In both formulas h is the increment in x. As extensions of those formulas to integration over intervals which are multiples of h in length there are:

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[f(x) + f(x_n) \right] + h\sum_{j=1}^{n-1} f(x_j) - \frac{nf''(s)h^3}{12} \tag{46}$$

where $x_0 < s < x_n$, and

$$\int_{x_0}^{x_{2n}} f(x)dx = \frac{h}{3}\left[f(x_0) - f(x_{2n}) \right] + \frac{h}{3}\sum_{j=1}^{\frac{n}{2}} \left[4f(x_{2j-1}) + 2f(x_{2_j}) \right] - \frac{n\, f^{(iv)}(s)h^5}{90} \tag{47}$$

where $x_0 < s < x_{2n}$.

Consider now the integration of

$$I = \int_{t_{m-1}}^{t_m} \int_{x_0}^{x_N} v(x,t)dx\,dt \quad , \tag{48}$$

where $v(x,t)$ is assumed to be of class C^4. Let the interval $[\ x_0\ ,\ x_N\ ]$ be divided into N equal subintervals of length h, and let $t_m - t_{m-1} = k > 0$. Integrating first with respect to x by means of (46) gives:

$$I = \int_{t_{m-1}}^{t_m} h\left[\tfrac{1}{2}\,v(x_0,t) + v(x_1,t) + \ldots + \tfrac{1}{2}\,v(x_N t)\right]dt$$

$$- \int_{t_{m-1}}^{t_m} \frac{Nh^3}{12}\,\frac{\partial^2 v(s,t)}{\partial x^2}\,dt, \tag{49}$$

where $x_0 < s < x_N$. Integrating now with respect to t by means of (44) gives:

$$I = \frac{hk}{2}\left(\Sigma_{m-1} + \Sigma_m\right) - \frac{hk^3}{24}\,\frac{\partial^2 v(x_0 z_0)}{\partial t^2} + \frac{\partial^2 v(x_0 z_N)}{\partial t^2}$$

$$- \frac{hk^3}{12}\sum_{j-1}^{N-1}\frac{\partial^2 v(x_j z_j)}{\partial t^2} - \frac{Nh^3 k}{24}\,\frac{\partial^2 v(s,t_{m-1})}{\partial x^2} + \frac{\partial^2 v(x,t_m)}{\partial x^2}$$

$$+ \frac{Nh^3 k^3}{144}\,\frac{\partial^4 v(s,z_{N+1})}{\partial t^2\,\partial x^2} \quad , \tag{50}$$

where:

$$\Sigma_j = \frac{1}{2} v(x_0, t_j) + v(x_1, t_j) + \ldots + v(x_{N-1}, t_j) + \frac{1}{2} v(x_N, t_j) \ ,$$

for $j = m-1,\ m$;

$t_{m-1} < z_j < t_m$ for $j = 0, \ldots, N+1$;

$x_0 < s < x_N$.

The truncation error terms may be put in more concise form by applying the intermediate value theorem:

$$I = \frac{hk}{2} \left(\Sigma_{m-1} + \Sigma_m \right) - \frac{Nhk^3}{12} \frac{\partial^2 v(s_1, z_1)}{\partial t^2} - \frac{Nh^3k}{12} \frac{\partial^2 v(s_2, z_2)}{\partial x^2}$$

$$+ \frac{Nh^3k^3}{144} \frac{\partial^4 v(s_3, z_3)}{\partial t^2 \partial x^4} \qquad (51)$$

where $x_0 < s_j < x_N$ for $j = 1, 2, 3$, and where $t_{m-1} < z_j < t_m$ for $j = 1, 2, 3$. Similar methods may be applied using the Simpson's rule formulas (45) and (47). Consider

$$I = \int_{t_{m-1}}^{t_{m+1}} \int_{x_0}^{x_{2n}} v(x,t)dx\, dt \quad , \qquad (52)$$

where $v(x,t)$ is assumed to be of class C^8, and let $[\, x_0, x_{2n}\,]$ be divided into 2n subintervals of equal length h; let

$[\, t_{m-1}, t_{m+1} \,]$ be divided into two intervals of length k. Integrating first with respect to x by means of (47) gives:

$$I = \int_{t_{m-1}}^{t_{m+1}} \frac{h}{3}\Big(v(x_0,t) + 4v(x_1,t) + \ldots. + v(x_{2n},t)\Big)\,dt$$

$$- \frac{nh^5}{90} \int_{t_{m-1}}^{t_m} \frac{\partial^4 v(s,t)}{\partial x^4}\,dt \quad , \qquad (53)$$

where $x_0 < s < x_{2n}$. Integrating with respect to t by means of (45) then yields:

$$I = \frac{hk}{9}\left(\Sigma_{m-1} + 4\Sigma_m + \Sigma_{m+1}\right)$$

$$- \frac{nh^5k}{270}\left(\frac{\partial^4 v(s,t_{m-1})}{\partial x^4} + 4\frac{\partial^4 v(s,t_m)}{\partial x^4} + \frac{\partial^4 v(s,t_{m+1})}{\partial x^4}\right)$$

$$- \frac{hk^5}{270}\left(\frac{\partial^4 v(x_0,z_0)}{\partial t^4} + 4\,\frac{\partial^4 v(x_1,z_1)}{\partial t^4} + \ldots. + \frac{\partial^4 v(x_{2n},z_{2n})}{\partial t^4}\right)$$

$$+ \frac{nh^5k^5}{90^2}\,\frac{\partial^8 v(s,z_{2n+1})}{\partial t^4\,\partial x^4} \qquad (54)$$

where

$$\Sigma_j = v(x_0,t_j) + 4v(x_1,t_j) + 2v(x_2,t_j) + \ldots + 4v(x_{2n-1},t_j)$$

$$+ v(x_{2n},t_j) \qquad \text{for } j = m-1,\ m,\ m+1.$$

By means of the intermediate value theorem

$$\frac{\partial^4 v(s,t_{m-1})}{\partial x^4} + 4\,\frac{\partial^4 v(s,t_m)}{\partial x^4} + \frac{\partial^4 v(s,t_{m+1})}{\partial x^4} = 6\,\frac{\partial^4 v(s_1,z_1)}{\partial x^4}\ , \tag{55}$$

for some z_1: $t_{m-1} < z_1 < t_{m+1}$; also

$$\frac{\partial^4 v(x_0,z_0)}{\partial t^4} + \frac{^4 v(x_{2n},z_{2n})}{t^4} = 2\,\frac{\partial^4 v(s^*,z^*)}{\partial t^4}\ ,$$

for some s^*, z^*: $x_0 < s^* < x_{2n}$; $z_0 < z^* < z_{2n}$. Then

$$2\,\frac{\partial^4 v(s^*,z^*)}{\partial t^4} + 4\frac{\partial^4 v(x_1,z_1)}{\partial t^4} + \ldots\ldots + 2\,\frac{\partial^4 v(x_{2n-2},z_{2n-2})}{\partial t^4}$$

$$+ 4\,\frac{\partial^4 v(s_{2n-1},z_{2n-1})}{\partial t^4} = 6n\,\frac{\partial^4 v(s,z)}{\partial t^4} \tag{56}$$

for some $x_0 < s < x_{2n}$, $t_{m-1} < z < t_{m+1}$. Using these expressions in (54) gives

$$I = \frac{hk}{9}\left(\Sigma_{m-1} + 4\Sigma_m + \Sigma_{m+1}\right) = \frac{nh^5k}{45}\ \frac{\partial^4 v(s_1,z_1)}{\partial x^4}$$

$$-\ \frac{nhk^5}{45}\ \frac{\partial^4 v(s_2,z_2)}{\partial t^4} + \frac{nh^5x^5}{90^2}\ \frac{\partial^8 v(s_3,z_3)}{\partial t^4\partial x^4} \tag{57}$$

where $x_0 < s_j < x_{2n}$ for $j = 1, 2, 3$, and where $t_{m-1} < z_j < t_{m+1}$ for $j = 1, 2, 3$.

Either formula (51) or (57) may be applied to the equation (43), and of course it is possible to develop formulas of higher order. However, in this respect it should be mentioned that higher order methods in the t - direction require, of course, more "profiles" of the temperature function, $T(x,t)$; for example using the "trapezoidal rule" for integration with respect to t only $T(x,t_j)$ and $T(x,t_{j+1})$ are needed; for "Simpson's rule" $T(x,t_{j-1})$, $T(x,t_j)$, and $T(x,t_{j+1})$ are needed; for a higher order process such as "Weddle's rule" seven temperature profiles would be required. In such a (higher order) procedure the special starting techniques required might well prove to be more cumbersome and time-consuming than would be worthwhile. For integration in the x - direction this problem is not present, and higher order approximation may easily be used. Of course it is also possible to use a combination of techniques in a given double integral problem. For example one could use the "trapezoidal rule" to evaluate the integral in the x - direction and "Simpson's rule" in the t - direction.

A numerical procedure for solving the entire system.--The procedure to be used for solving

$$\frac{\partial T(x,t)}{\partial t} = \frac{X}{\varepsilon} \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{n(x)}{\varepsilon} (P(t) - 1) \tag{58}$$

is obtained by taking n = 2 in equation (5) of Chapter II. Thus

$$T(x,t+k) = T(x,t) + k \frac{\partial T(x,t)}{\partial t} + \frac{k^2}{2} \frac{\partial^2 T(x,t)}{\partial t^2} + R_1(x,t) \tag{59}$$

where $R_1(x,t)$ is the error term in (5) for n = 2. It will be convenient to let $u(t) = \ln P(t)$. Assuming $T(x,t)$ to be of class C^4 and differentiating (58) with respect to t:

$$\frac{\partial^2 T(x,t)}{\partial t^2} = \frac{X}{\varepsilon} \frac{\partial^2}{\partial x^2} \left(\frac{\partial T(x,t)}{\partial t} \right) + \frac{n(x)}{\varepsilon} \frac{d(e^{u(t)}-1)}{dt} . \tag{60}$$

Substituting for $\frac{\partial T}{\partial t}$ from (58) gives:

$$\frac{\partial^2 T(x,t)}{\partial t^2} = \frac{X^2}{\varepsilon^2} \frac{\partial^4 T(x,t)}{\partial t^4} + \frac{X}{\varepsilon^2} (e^{u(t)}-1) \frac{d^2 n(x)}{dx^2}$$

$$+ \frac{n(x)}{\varepsilon} e^{u(t)} \frac{du(t)}{dt} . \tag{61}$$

Then replacing $\frac{\partial T}{\partial t}$ and $\frac{\partial^2 T}{\partial t^2}$ in (59) by means of (58) and (61):

$$T(x,t+k) = T(x,t) + \frac{kX}{\varepsilon} \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{k^2 X^2}{2\varepsilon^2} \frac{\partial^4 T(x,t)}{\partial x^4} + \frac{kn(x)}{\varepsilon} (e^{u(t)}-1)$$

$$+ \frac{k^2}{2\varepsilon} \left(\frac{X}{\varepsilon} (e^{u(t)}-1) \frac{d^2 n(x)}{dx^2} + n(x)\, e^{u(t)} \frac{du(t)}{dt} \right) + R_1(x,t) \tag{62}$$

If the central difference approximations

$$\frac{\partial^2 T(x,t)}{\partial x^2} \approx \frac{1}{h^2}\left(\delta_x^2 T(x,t) - \frac{1}{12}\delta_x^4 T(x,t)\right) \quad , \tag{63}$$

$$\frac{\partial^4 T(x,t)}{\partial x^4} \approx \frac{1}{h^4}\delta_x^4 T(x,t) \quad , \tag{64}$$

are used in (62) there results the following difference equation for $T(x,t)$:

$$T(x,t+k) = T(x,t) + \lambda\,\delta_x^2 T(x,t) - \frac{\lambda}{12}\delta_x^4 T(x,t)$$

$$+ \frac{\lambda^2}{2}\delta_x^4 T(x,t) + g(x,t) + R(x,t), \tag{65}$$

where $\lambda = \frac{kX}{h^2}$, $R(x,t)$ is the truncation error term, and $g(x,t)$ is the non-homogeneous part of (62)--that is, that part of (62) involving $u(t)$. In terms of values of $T(x,t)$ (65) becomes

$$T(x,t+k) = \left(\frac{\lambda^2}{2} - \frac{\lambda}{12}\right)\Big(T(x-2h,t) + T(x+2h,t)\Big)$$

$$+ \left(\frac{4\lambda}{3} - 2\lambda^2\right)\Big(T(x-h,t) + T(x+h,t)\Big)$$

$$+ \left(1 - \frac{5\lambda}{2} + 3\lambda^2\right)\Big(T(x,t)\Big) + g(x,t) + R(x,t) \,. \tag{66}$$

If $g(x,t)$ is even with respect to x, it is clear that $T(x,t+k)$ as computed by (66) will be an even function of x provided $T(x,t)$ is an even function of x. Henceforth it will be assumed:

$$\begin{aligned} T_0(x) &= T_0(-x) \quad , \\ g(x,t) &= g(-x,t) \quad , \qquad (67) \\ \alpha(x) &= \alpha(-x) \quad . \end{aligned}$$

Thus $T(x,t)$ will be an even function of x for all t, and equation (43) takes the form:

$$u(t) = u(t_0) - 2 \int_{t_0}^{t} \int_{0}^{\infty} \alpha(x)\, T(x,s)\, dx\, ds \quad , \qquad (68)$$

where $u(t) = \ln P(t)$. Let N_m be that positive integer for which it is true that

$$\left| \sum_{j=N_m+1}^{\infty} \alpha(x_j)\, T(x_j,t_m) \right| < 10^{-a} \quad , \qquad (69)$$

where a is the number of decimal places to be used in the computation. Equation (68) can then be approximated in the following manner using equation (51):

$$u(t_m) = u(t_{m-1}) - hk\, (\Sigma_{m-1} + \Sigma_m) + E_1(x,t) \quad , \qquad (70)$$

where*

$$\Sigma_m = \tfrac{1}{2}\alpha_0 T_{0m} + \alpha_1 T_{1m} + \dots + \alpha_{N_m-1} T_{N_m-1,m} + \tfrac{1}{2}\alpha_{N_m} T_{N_m,m} \ ,$$

with N_m determined by (69); and where

$$E_1(x,t) = -\frac{N_m hk}{6}\left(k^2 \frac{\partial^2 \alpha T}{\partial t^2} + h^2 \frac{\partial^2 \alpha T}{\partial x^2}\right) + \frac{N_m h^3 k^3}{72} \frac{\partial^4 \alpha T}{\partial t^2 \partial x^2} \ . \qquad (71)$$

In a similar fashion (68) can be approximated using the equation (57):

$$u_{m+1} = u_{m-1} - \frac{2hk}{9}\left(\Sigma_{m-1} + 4\Sigma_m + \Sigma_{m+1}\right) + E_2(x,t) \quad , \qquad (72)$$

where

$$\Sigma_m = \alpha_0 T_{0m} + 4\alpha_1 T_{1m} + 2\alpha_2 T_{2m} + \dots + 4\alpha_{N_m-1,m} T_{N_m-1,m}$$

$$+ \alpha_{N_m} T_{N_m,m} \quad ,$$

and where

$$E_2(x,t) = -\frac{2N_m hk}{45}\left(k^4 \frac{\partial^4 \alpha T}{\partial t^4} + h^4 \frac{\partial^4 \alpha T}{\partial x^4}\right)$$

$$+ \frac{2N_m h^5 k^5}{90^2} \frac{\partial^8 \alpha T}{\partial t^4 \partial x^4} \ . \qquad (73)$$

---

*In order to simplify the notation, f_{jm} will be used for the quantity $f(x_j, t_m)$.

Equation (66) in combination with (70) or (72) provides a method for solving numerically the system:

$$\frac{du(t)}{dt} = -2 \int_0^{\infty} \alpha(x)\ T(x,t)\ dx \tag{74}$$

$$\frac{\partial T(x,t)}{\partial t} = \frac{\chi}{\varepsilon} \ \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{n(x)}{\varepsilon} \ (e^{u(t)}-1) \quad , \tag{75}$$

with the initial conditions:

$$u(0) = u_0 \quad , \tag{76}$$

$$T(x,0) = T_0 \quad , \tag{77}$$

where it is assumed that $\alpha(x)$, $n(x)$, $T_0(x)$ are even functions of x. In the particular cases considered for actual computation in the next chapter there is no question about existence of the integral in (74). Otherwise additional hypothesis would be needed. Here, then, is a complete algorithm; call it Method A:

(a) Choose h, the increment in x, and k, the increment in t, so as to achieve the desired accuracy. In order to start the computation it is also necessary to choose a k^* such that $nk^* = k$, n a positive integer greater than one.

(b) Using equation (66) compute $T(jh,(m^* + 1)k^*)$ for $j = 0, 1, \ldots, N$. N is that positive even integer such that

$$\left| \sum_{j=N+1}^{\infty} \alpha_j T_{j,(m^*+1)k} \right| < 10^{-a} \quad . \qquad (78)$$

In (66) $\lambda = \frac{k^* X}{h^2}$. For the first step the initial values u_0 and $T_0(x)$ are used.

(c) Using equation (70) compute $u(m^*+1)k^*)$. For $m^* = 0$ the given initial values u_0 and $T_0(x)$ are used.

(d) Repeat (b) and (c) until $T(jh, nk^*) = T(jh, k)$ and $u(nk^*) = u(k)$ have been computed.

(e) Using equation (66) compute $T(jh,(m+1)k)$ as in step (a), where now $\lambda = \frac{kX}{h^2}$. For the first step the values $u(k)$ and $T(jh,k)$ as given by steps (b) - (c) are used.

(f) Using equation (72), compute $u((m+1)k)$. For computing $u(2k)$ the given initial values u_0, $T_0(jh)$ as well as the values $T(jh,k)$ and $T(jh,2k)$ are used.

(g) Repeat steps (e) and (f) as many times as desired. At each step the values for $t = mk$ replace those for $t = (m-1)k$, and those for $t = (m+1)k$ replace those for $t = mk$.

<u>Remarks</u>.

(i) The trapezoidal formula (70) must be used to start the procedure. This is true since initially only two profiles of $T(x,t)$--namely $T_0(x)$ and $T(x,k)$--are available, and formula (72) requires three profiles. A convenient choice of k^* is $k^* = (0.01)k$ and this provided sufficient accuracy for the problems solved in Chapter IV.

(ii) Clearly it is not necessary to retain three (or two) complete profiles of $T(x,t)$; only the sums of these values as indicated by equation (70) or (72) need be saved in order to advance the calculation.

(iii) The number N as determined by (78) depends of course on the accuracy desired in $u(t)$. It is desirable to have N fixed for the entire calculation and this is indeed possible in many applications where properties of $\alpha(x)$ and $T(x,t)$ allow a choice of N which will be acceptable for all values of t. It is clearly no restriction to assume that N is even so that it will apply to the Simpson's rule formula.

<u>Error and stability analysis for Method A</u>.--In order to compute the truncation error term $R(x,t)$ of equation (66) it is convenient to put that equation in the form:

$$T(x,t+k) = T(x,t) + \lambda\delta_x^{\,2}T(x,t) + \left(\frac{\lambda^2}{2} - \frac{\lambda}{12}\right)\delta_x^{\,4}T(x,t)$$

$$+ \frac{kn}{\varepsilon}\left(e^{u(t)}-1\right) + \frac{k^2}{2}f(x,t) + R(x,t) \tag{79}$$

where $f(x,t)$ is the non-homogeneous term in (62) involving derivatives of $n(x)$ and $u(t)$. It is easily shown that (cf. Milne-Thomson [14])

$$\delta^2 T(x,t) = h^2\,\frac{\partial^2(x,t)}{\partial x^2} + \frac{h^4}{12}\,\frac{\partial^4 T(x,t)}{\partial x^4} + \frac{h^6}{360}\,\frac{\partial^6 T(s_1,t)}{\partial x^6}\,, \tag{80}$$

$$\delta^4 T(x,t) = h^4\,\frac{\partial^4 T(x,t)}{\partial x^4} + \frac{h^6}{6}\,\frac{\partial^6 T(s_2,t)}{\partial x^6}\,, \tag{81}$$

for some s_1 and s_2 on the open interval $(x-2h,\ x+2h)$. Thus on substitution of (80) and (81) in (79) and rearranging terms

$$\frac{T(x,t+k)-T(x,t)}{k} = \frac{X}{\varepsilon}\,\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{n(x)}{\varepsilon}\left(e^{u(t)}-1\right) + \frac{kX^2}{2\varepsilon^2}\,\frac{\partial^4 T(x,t)}{\partial x^4}$$

$$+ \frac{h^4X}{6\varepsilon}\left(\frac{\lambda}{2} - \frac{1}{12}\right)\frac{\partial^6 T(s_2,t)}{\partial x^6} + \frac{h^4X}{360\varepsilon}\,\frac{\partial^6 T(s_1 t)}{\partial x^6}$$

$$+ \frac{k}{2}f(x,t) + R(x,t)\quad . \tag{82}$$

Now by Taylor's theorem with remainder,

$$\frac{T(x,t+k)-T(x,t)}{k} = \frac{\partial T(x,t)}{\partial t} + \frac{k}{2}\,\frac{\partial^2 T(x,t)}{\partial t^2} + \frac{k^2}{6}\,\frac{\partial^3 T(x,z)}{\partial t^3}\quad , \tag{83}$$

for some $z : t < z < t+k$. Next substitute for $\frac{\partial^2 T}{\partial t^2}$ in (83) by means of (61):

$$\frac{T(x,t+k)-T(x,t)}{k} = \frac{\partial T(x,t)}{\partial t} + \frac{kX^2}{2\varepsilon^2}\frac{\partial^4 T}{\partial x^4} + \frac{k}{2} f(x,t) + \frac{k^2}{6}\frac{\partial^3 T(x,t)}{\partial t^3} \quad (84)$$

for some $z : t < z < t+k$. On substituting (84) in (82) one obtains:

$$\frac{\partial T(x,t)}{\partial t} - \frac{X}{\varepsilon}\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{n(x)}{\varepsilon}\left(e^{u(t)}-1\right) = \frac{k^2}{6}\frac{\partial^3 T(x,z)}{\partial t^3}$$
$$+ \frac{h^4 X}{360\varepsilon}\frac{\partial^6 T(s_1,t)}{\partial x^6} + \frac{h^4 X}{6\varepsilon}\left(\frac{\lambda}{2} - \frac{1}{12}\right)\frac{\partial^6 T(s_2,t)}{\partial x^6} + R(x,t) \quad (85)$$

Now the right hand side of (85) is zero by equation (58) and thus (applying the intermediate value theorem to the sixth order derivatives with respect to x):

$$R(x,t) = -\frac{h^4 X}{\varepsilon}\left[\left(\frac{\lambda}{12} - \frac{1}{90}\right)\frac{\partial^6 T(s,t)}{\partial x^6} + \frac{\lambda^2\varepsilon^3}{6X^3}\frac{\partial^3 T(x,z)}{\partial t^3}\right] \quad (86)$$

for some $s : x-2h < s < x+2h$, and some $z : t < z < t+k$. Although (86) is the usual form of giving the truncation error whereby one normally states that $R(x,t)$ is $O(h^4)$, it is better for the purposes of what is to follow, using

$\lambda = \frac{X k}{h^2}$, to write:

$$R(x,t) = - h^2 k \frac{X^2}{12\,\varepsilon^2} \frac{\partial^6 T(s_1,t)}{\partial x^6} - \frac{k^2}{6} \frac{\partial^3 T(x,z)}{\partial t^3} + \frac{h^4 X}{90\varepsilon} \frac{\partial^6 T(s_2,t)}{\partial x^6} .$$

The above result may be stated as the following:

Theorem 8. If $T(x,t)$ is of class C^6 in R then the truncation error $R(x,t)$ of equation (79) is given by:

$$R(x,t) = -h^2 k \frac{X^2}{12\varepsilon^2} \frac{\partial^6 T(s_1,t)}{\partial x^6} - \frac{k^2}{6} \frac{\partial^3 T(x,z)}{\partial t^3} + \frac{h^4 X}{90\varepsilon} \frac{\partial^6 T(s_2,t)}{\partial x^6} \quad (87)$$

for some s_1, s_2 and z: $t < z < t+k$, $x-2h < s_1, s_2 < x+2h$.

There is an additional error involved in computation of $T(x,t)$ by (66). This error is that due to errors in the values used for $u(t)$ and $\frac{du(t)}{dt}$ in the non-homogeneous terms. In view of the results of Chapter II the equation (66) will be stable provided the coefficients C_r satisfy the requirements of Theorem 4 or of Theorem 5. The crucial point, however, is this: Even though errors in $u(t)$ and $\frac{du(t)}{dt}$ cannot *per se* affect the stability of equation (66), they may cause error build-up in the numerical results in just the same fashion as round-off errors do. In fact, if the error in computing $u(t)$ is such that only p digits of accuracy are obtained and $T(x,t)$ is roughly of the same order of magnitude as the non-homogeneous terms, then one is effectively computing $T(x,t)$ in a p-digit computing device. No matter how small the truncation error $R(x,t)$ is made no increase in accuracy of $T(x,t)$ can result unless a corresponding increase in accuracy of $u(t)$ is effected.

The stability of equation (66) will be proved as the following:

Theorem 9. If $g(x,t)$ and $T_0(t)$ are bounded in $R : -\infty < x < \infty,\ 0 \leq t \leq tn$, and if: $0 < \lambda \leq 2/3$, then equation (66) is stable.

Proof: Consider the coefficients c_r of equation (66):

$$c_{-2} = c_2 = \frac{\lambda^2}{2} - \frac{\lambda}{12}$$

$$c_{-1} = c_1 = \frac{4\lambda}{3} - 2\lambda^2$$

$$c_0 = 1 + 3\lambda^2 - 5\lambda/2$$

Since they are all constants they are certainly uniformly bounded in R. Now,

$$\sum_r c_r(x,t) = 1 + 3\lambda^2 - \frac{5\lambda}{2} + 2\left(\frac{4\lambda}{3} - 2\lambda^2\right) + 2\left(\frac{\lambda^2}{2} - \frac{\lambda}{12}\right)$$

$$= 1 + \left(-5/2 + 8/3 - 1/6\right)\lambda + (3 - 4 + 1)\lambda^2 = 1$$

$$\sum_r rc_r(x,t) = 12c_{-2} - c_{-1} + c_1 + 2c_2 = 0$$

$$\sum_r r^2 c(x,t) = 4c_{-2} + c_{-1} + c_1 + 4c_2 = 8c_2 + 2c_1$$

$$= 4\lambda^2 - 2\lambda/3 + 8\lambda/3 - 4\lambda^2 = 2\lambda = \frac{2k}{h^2}\ \frac{X}{\varepsilon}$$

Thus the c_r satisfy the relations (14). Now,

$$c_{-2},\ c_2 \geq 0 \Rightarrow \frac{\lambda^2}{2} - \frac{\lambda}{12}\ \_\ 0 \Rightarrow \frac{\lambda}{2} \geq \frac{1}{12}, \text{ or } \lambda \geq 1/6$$

$$c_{-1},\ c_1 \geq 0 \Rightarrow \frac{4\lambda}{3} - 2\lambda^2 \geq 0 \Rightarrow 2\lambda \leq 4/3, \text{ or } \leq 2/3 .$$

C_0 is positive for all real values of λ. Thus by Theorem 5 the equation (66) is stable for $1/6 \leq \lambda < 2/3$. In order to show the stability of (66) for λ in the range $0 < \lambda < 1/6$ Theorem 4 will be used. For this it is sufficient to show that the relation (41) is satisfied, since the other requirements of the theorem are clearly satisfied,

$$\sum_r c_r e^{ir\theta} = c_{-2}\, e^{-2i\theta} + c_{-1}\, e^{-i\theta} + c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta}$$

$$= 2c_2 \cos 2\theta + 2c_1 \cos\theta + c_0 = g(\theta) \qquad (88)$$

It is to be shown that there exists some $M > 0$ such that $|g(\theta)| \leq e^{-M\theta^2}$ for $|\theta| \leq \pi$. Since $g(\theta)$ is an even function of θ it will be sufficient to consider the closed interval $[0,\pi]$. Also, $g(\theta) \geq 0$ for $0 \leq \theta \leq \pi$, for the minimum value possible for $0 < \lambda < 1/6$ is: $c_0 - 2c_1 - 2|c_2|$

which is positive if

$$c_0 \geq 2c_1 + 2|c_2| \quad ,$$

i.e. if

$$1 - \frac{5\lambda}{2} + 3\lambda^2 \geq \frac{8\lambda}{3} - 4\lambda^2 + \frac{\lambda}{6} - \lambda^2 ,$$

which holds if,

$$1 \geq \frac{16\lambda}{3} - 8\lambda^2 \quad ,$$

i.e. if

$$1 \geq \frac{16\lambda}{3}$$

which is true for $\lambda \leq \frac{3}{16}$. But $\lambda < 1/6 < 3/16$ and hence $g(\theta) \geq 0$ for $0 \leq \theta \leq \pi$. Thus it suffices to prove that $g(\theta) \leq e^{-M\theta^2}$ for $0 \leq \theta \leq \pi$ and suitable M.

Now on the interval $0 \leq \theta \leq \pi/4$ it is true that

$$0 \leq \cos\theta \leq 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} \quad ,$$

and

$$1 \geq \cos 2\theta \geq 1 - \frac{4\theta^2}{2!} \quad .$$

Hence

$$g(\theta) \leq c_0 + 2c_1 \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}\right) + 2c_2 \left(1 - \frac{4\theta^2}{2}\right) ,$$

or equivalently

$$g(\theta) \leq (c_0 + 2c_1 + 2c_2) - (c_1 + 4c_0)\, \theta^2 + \frac{c_1}{12}\, \theta^4$$

or, substituting for c_0, c_1, c_2,

$$g(\theta) \leq 1 - \lambda\theta^2 + \frac{c_1}{12}\, \theta^4 = 1 - \left(\lambda - \frac{c_1}{12}\, \theta^2\right) \theta^2 \quad .$$

Now

$$\lambda - \frac{c_1}{12}\, \theta^2 = \lambda - \left(\frac{\lambda}{9} - \frac{\lambda^2}{12}\right)\theta^2 \geq \frac{8\lambda}{9} + \frac{\lambda^2}{12} = M_1$$

for $0 \leq \theta \leq \pi/4$. Hence

$$g(\theta) \leq 1 - M_1\theta^2 \leq e^{-M_1\theta^2} \quad \text{for } 0 \leq \theta \leq \pi/4 \quad . \qquad (89)$$

Now $g'(\theta) = -2c_1 \sin \theta - 4c_2 \sin 2\theta$, and $g'(\theta)$ will be less than zero on $(0, \pi)$ provided

$$2c_1 \sin \theta > 4|c_2| \sin 2\theta \quad ,$$

i.e. if

$$\frac{c_1}{2\,|c_2|} > 2 \cos \theta \quad .$$

But $\frac{c_1}{2|c_2|} > 2$ as may easily be seen by comparison of the expression for c_1 and $|c_2|$. Thus $g(\theta)$ is monotonic decreasing on $\frac{\pi}{4} < \theta \leq \pi$. Choose M_2 such that $e^{-M_2 \; 2} = g(\pi/4)$, and let $M = \min(M_1, M_2)$. Then

$$g(\theta) \leq e^{-M\theta^2} \quad \text{for } 0 \leq \theta \leq \pi \; .$$

Thus equation (66) is stable for $0 < \lambda < {}^1/6$ by Theorem 4, and thus the total range of λ for which (66) is stable is: $0 < \lambda \leq {}^2/3$.

The truncation error in computing u(t) due to the Simpson's rule approximation is given by (73); that due to the values of T(x,t) is given by (87). The total truncation error, T.E., committed in the evaluation of (72) is given by:

$$\begin{aligned} \text{T.E.} = \frac{2hk}{9} [&R(0,t_{m-1}) + 4R(h,t_{m-1}) + 2R(2h,t_{m-1}) + \cdots \\ &\cdots + R(Nh,t_{m-1})] + 4[R(0,t_m) + 4R(h,t_m) + \cdots \\ &\cdots + R(Nh,t_m)] + [R(0,t_{m+1}) + \cdots \\ &\cdots + R(Nh,t_m)] + E_2(x,t) \quad , \end{aligned} \tag{90}$$

where $E_2(x,t)$ is given by (73) and $R(x,t)$ is given by (87). Now from equation (87),

$$\left|R(x,t)\right| \leq h^2kM_1 + k^2M_2 + h^4M_3 \tag{91}$$

for (x,t) in the region S:

$$0 \leq x \leq Nh$$

$$t_{m-1} \leq t \leq t_{m+1} \quad ,$$

and where

$$M_1 = \underset{S}{\text{l. u. b.}} \left| \frac{X^2}{12\varepsilon^2} \frac{\partial^6 T(x,t)}{\partial x^6} \right| ,$$

$$M_2 = \underset{S}{\text{l. u. b.}} \left| \frac{1}{6} \frac{\partial^3 T(x,t)}{\partial t^3} \right| ,$$

$$M_3 = \underset{S}{\text{l. u. b.}} \left| \frac{X}{90\varepsilon} \frac{\partial^6 T(x,t)}{\partial x^6} \right| .$$

From (73),

$$\left| E_2(x,t) \right| \leq \frac{2Nhk}{45} (k^4M_4 + h^4M_5) + \frac{2Nh^5k^5}{90^2} M_6 \tag{92}$$

for (x,t) in S, and where:

$$M_4 = \underset{S}{\text{l. u. b.}} \left| \frac{\partial^4 T}{\partial t^4} \right| ,$$

$$M_5 = \underset{S}{\text{l. u. b.}} \left| \frac{\partial^4 T}{\partial x^4} \right| ,$$

$$M_6 = \underset{S}{\text{l. u. b.}} \left| \frac{\partial^6 T}{\partial x^6} \right| .$$

Thus,

$$|\text{T.E.}| \leq \frac{2hk}{9} [3N(h^2kM_1 + k^2M_2 + k^4M_3) + 12N(h^2kM_1 + k^2M_2 + h^4M_3) + 3N(h^2kM_1 + k^2M_2 + k^4M_3)] + \frac{2Nhk}{45} (k^4M_4 + h^4M_5) + \frac{2Nh^5k^5}{90^2} M_6 ,$$

or equivalently,

$$|\text{T.E.}| \leq 4Nhk^3M_2 + 4Nh^3k^2M_1 + 4h^5kN \left(M_3 + \frac{M_5}{90} \right) + \frac{2Nhk^5}{45} M_4 + \frac{2Nh^5k^5}{90^2} M_6 . \tag{93}$$

In (93) note that the terms of lowest order in h and k--that is, the terms of greatest magnitude--are those terms arising from the truncation errors in $T(x,t)$. Moreover, note that in the first term k appears to the third power while h appears linearly; thus one would expect adjustment of the time step to affect more readily the accuracy of the results.

It has already been noted that errors in $u(t)$ cause errors in $T(x,t)$ in addition to the truncation errors of the equation (66). At the next step of the computation these additional errors in $T(x,t)$ enter the computation of $u(t)$. Thus while there is no concern over stability--in the sense given in the introduction--when performing a normal numerical integration, the integration here is a stage-wise process in which errors committed at one step reenter at the following step. That is, normally one assumes that all the values to be used in the integrand are known beforehand; here the values to be used must be computed step-by-step. Thus if the computation of the values of the integrand is an unstable process then the numerical integration is itself "unstable."

Such is not the case here, since the procedure to be used in calculating $T(x,t)$ has been shown to be stable. The interaction of errors in the computational procedure, however, is of considerable interest in obtaining accurate results and will be considered more explicitly for the special case to be given in Chapter IV.

CHAPTER IV

NUMERICAL EXAMPLES

A special case of the general systems.--Under the assumptions that X = 0 and α is a constant the system (1) becomes:

$$\frac{d\ln P(t)}{dt} = -\alpha \int_{-\infty}^{\infty} T(x,t)dx \tag{94}$$

$$\varepsilon \frac{T(x,t)}{t} = n(x)\ (P(t) - 1) \tag{95}$$

where $T(x,0) = T_0(x)$ and $P(0) = P_0$ are the given initial data. For P(t) and T(x,t) as given by (94) and (95) there is the following

Theorem 10. If α and ε are positive constants and $n(x) > 0$ then the system consisting of (94) and (95) possesses solutions P(t) and T(x,t) which are periodic with respect to t.

Proof: Let

$$g(t) = \int_{-\infty}^{\infty} T(x,t)dx \quad ; \tag{96}$$

then

$$\frac{\varepsilon\, dg(t)}{dt} = \int_{-\infty}^{\infty} \frac{\varepsilon T(x,t)}{t}\, dx \quad ,$$

or

$$\varepsilon \frac{dg(t)}{dt} = (P(t) - 1) \int_{-\infty}^{\infty} n(x)dx \quad . \tag{97}$$

Now $\int_{-\infty}^{\infty} n(x)dx = 1$ since $n(x)$ represents the fraction of the power extracted at x. Thus,

$$\varepsilon \frac{dg(t)}{dt} = P(t) - 1 \tag{98}$$

and the original system is replaced by:

$$\frac{d\ln P(t)}{dt} = -\alpha g(t)$$

$$\frac{dg(t)}{dt} = P(t) - 1 \quad .$$

This system is of the same form as the one given by Ergen, Lipkin and Nohel [15] which describes the dynamics of a reactor with constant power extraction. Hence by <u>Theorem 1</u> of that paper P(t) and g(t) are periodic in t if α , ε , n > 0 ; in fact, as shown in [15]:

$$P(t) - \ln P(t) + \frac{\alpha\varepsilon}{2} g^2(t) = \text{constant}. \tag{99}$$

Now P(t) and g(t) periodic in t implies that T(x,t) is periodic in t. This is what was to be shown.

In view of the remarks on dynamic stability given in the introduction, this special case, then, provides a simple example of a dynamically stable reactor which may be used to check the numerical procedure outlined in Chapter III. In particular, it will be most useful for studying the interaction of errors between equations (66) and (72).

A numerical example for the special case.--Consider the case when:

$$
\begin{aligned}
\alpha(x) &= 1 & |x| &\leq 50 \\
\alpha(x) &= 0 & |x| &> 50 \\
n(x) &= 0.01 & |x| &\leq 50 \\
n(x) &= 0 & |x| &> 50 \\
\varepsilon &= 1/25 \\
X &= 0 \\
T_0(x) &\equiv 0 \\
\ln P(0) &= u_0 = 0.5
\end{aligned}
\tag{100}
$$

Equations (94) and (95) now take the forms:

$$\frac{d\,u(t)}{d} = -100\ T(t) \quad , \tag{101}$$

and

$$\frac{1}{25} \quad \frac{d\,T(t)}{dt} = (0.01)\,(e^{u(t)} - 1) \quad , \tag{102}$$

where $u(t) = \ln P(t)$. Note that since $n(x)$ and $T_0(x)$ were assumed to be constants on the interval: $-50 \leq x \leq 50$, $T(x,t)$ is now a function of t alone on this interval. The assumptions $n(x) = 0$, $|x| > 50$ and $T_0(x) \equiv 0$ determines that T is identically zero for $|x| > 50$.

From (101) it is clear that u(t) has a relative extremum when T(t) is zero, and likewise, from (102), that T(t) has a relative extremum when u(t) is zero. Thus, by choosing $T(0) = 0$, u_0 is the maximum value of u(t). The functions u(t) and T(t) satisfy the relation

$$e^{u} - u + (200)T^2 = (e^{u_0} - u_0) \simeq 1.1487 \tag{103}$$

as may easily be seen from equation (99) where the constant is evaluated from the given initial conditions:

$$T_o = 0$$

$$u_0 = 0.5 \quad .$$

The relation (103) provides a simple check on the numerical results which will be obtained.

The numerical procedure for solving (101) and (102) is a special case of (66) and (72). For n = 3 equation 5 of Chapter II now takes the form:

$$T(t+k) = T(t) + k\,\frac{dT(t)}{dt} + \frac{k^2}{2}\;\frac{d^2T(t)}{dt^2} + \frac{k^3}{6}\;\frac{d^3T(t)}{dt^3}$$

$$+ \frac{d^4T(z)}{dt^4} \qquad (104)$$

for some z: $t < z < t + k$. The derivatives of T(t) may be obtained from equation (102) by successive differentiation. Thus,

$$\frac{d^2T(t)}{dt^2} = (0.25)e^{u(t)}\;\frac{du(t)}{dt} \quad ,$$

$$\frac{d^3T(t)}{dt^3} = (0.25)e^{u(t)}\left(\frac{du^2}{dt} + \frac{d^2u}{dt^2}\right) \quad ,$$

$$\frac{d^4T(t)}{dt^4} = (0.25)e^{u(t)}\left(\frac{du^3}{dt} + 3\,\frac{d^2u}{dt^2}\,\frac{du}{dt} + \frac{d^3u}{dt^3}\right) \quad .$$

The derivatives of u(t) in these expressions are obtained from equation (101). Substitution for these yields

$$\frac{d^2T(t)}{dt^2} = (0.25)e^{u(t)}T(t) \quad ,$$

$$\frac{d^3T(t)}{dt^3} = (0.25)e^{u(t)}\left[\,10^4T^2(t) - 25(e^{u(t)}-1)\,\right] \quad ,$$

$$\frac{d^4T(t)}{dt^4} = (2500)\ T(t)e^{u(t)}\ [\ -10T^2(t) + (e^{u(t)}-.75)\ ]\ .$$

Thus,

$$T(t+k) = T(t) + (0.25)\ k(e^{u(t)}-1) - \frac{25k^2}{2}\ e^{u(t)}T(t)$$

$$+ \frac{25k^3}{6}\ e^{u(t)}\ [\ (10^2T^2(t) - (0.25)(e^{u(t)}-1)\ ]\ , \qquad (105)$$

where the truncation error is $O(k^4)$. Now from (101),

$$u(t+k) = u(t-k) - 100 \int_{t-k}^{t+k} T(t)dt \quad ,$$

and using Simpson's rule:

$$u(t+k) = u(t-k) - \frac{100k}{3}\ [\ T(t-k) + 4T(t) + T(t+k)\ ]\ , \qquad (106)$$

where the truncation error is $\frac{k^5}{90}\ \frac{d^4T(z)}{dt^4}$ for some z:

$t-k < z < t+k$.

In order to start the calculation it is necessary to use trapezoidal integration in the power equation. Initially, then, the following equation will be used.

$$u(t+k) = u(t) - 50k\ [\ T(t) + T(t+k)\ ] \quad , \qquad (107)$$

where the truncation error is $\frac{k^3}{12}\ \frac{d^3t(z)}{dt^3}$, $t < z < t+k$.

In order to simplify the analysis of error growth in the computation, the equations

$$u_{m+1} = u_{m-1} - \frac{100k}{3} [T_{m-1} + 4T_m + T_{m+1}] \tag{108}$$

$$T_{m+1} = T_m + 0.25k(e^{u_m} - 1) \tag{109}$$

will be considered. Equation (108) is the same as (106), but (109) is obtained from (105) by taking only two terms on the right-hand side of that equation. Similar, but of course, much more laborious analysis, may be applied in case higher order terms of (105) are taken. The idea here is to simply get an insight into the interaction of errors in the system.

Let v_m be an approximate value of u_m as given by (108). Let w_m be an approximate value of T_m as given by (109). Then

$$v_m = u_m + E_m \quad , \tag{110}$$

$$w_m = T_m + R_m \quad , \tag{111}$$

where E_m and R_m are the error terms. Substitution of (110) in (108) and (111) in (109) yields:

$$u_{m+1} + E_{m+1} = u_m + E_m - \frac{100k}{3} [(T_{m-1} + R_{m-1}) + 4 (T_m + R_m) + (T_{m+1} + R_{m+1})] \tag{112}$$

$$T_{m+1} + R_{m+1} = T_m + R_m + 0.25k\,(e^{u_m+E_m} - 1) \tag{113}$$

By subtracting (108) from (112) and (109) from (113) it is easily seen that:

$$E_{m+1} = E_m - \frac{100k}{3}\,(R_{m-1} + 4R_m + R_{m+1}) \quad , \tag{114}$$

and

$$R_{m+1} = R_m + 0.25k\,e^{u_m}\,(e^{E_m} - 1) \quad . \tag{115}$$

Note that while equation (114) is the same as (108), the same is not true of (115) and (109); this is due to the non-linear property of (109).

Now let:

$$R_0 = R_1 = R_2 = R > 0$$

$$E_0 = -\frac{100k}{3}\,R$$

$$E_1 = \frac{500k}{3}\,R \quad .$$

Then from (114)

$$E_2 = -\frac{100k}{3}\,R - \frac{100k}{3}\,[\,6R\,] = -\frac{700k}{3}\,R \quad .$$

Now

$$R_3 = R + 0.25k\,e^{u_2}\,(e^{-\frac{700k}{3}R} - 1) \quad .$$

Since R $>$ 0 this is equivalent to:

$$R_3 = R - 0.25k\, e^{u_2} \left(1 - e^{-\frac{700k}{3} R}\right) \quad ;$$

where $1 - e^{-\frac{700k}{3} R}$ is greater than zero but less than one. Hence R_3 will be less than R but greater than or equal to zero if and only if

$$0.25k\, e^{u_2} \left(1 - e^{-\frac{700k}{3} R}\right) \leq R \quad ,$$

or equivalently

$$1 - \frac{R}{0.25k\, e^{u_2}} \leq e^{-\frac{700k}{3} R} \quad .$$

This in turn will be true if

$$1 - \frac{R}{0.25k\, e^{u_2}} \leq 1 - \frac{700k}{3} R \quad ,$$

or if

$$\frac{700k}{3} R \leq \frac{R}{0.25k\, e^{u_2}} \quad .$$

This condition will be met if

$$k^2 \leq \frac{3}{(7)(25)e^{u_2}} \quad .$$

Knowing bounds on u(t) would then allow selection of a k sufficient to insure that $0 \leq R_3 < R$. Suppose this has been done and $R_3 = a_2R$ where $0 \leq a_2 < 1$. Then

$$E_3 = -\frac{500k}{3} - \frac{100k}{3} \quad (R + 4R + a_2R)$$

Thus

$$0 > E_3 > -\frac{1100k}{3} R.$$

Let $R_3 = R$ and $E_3 = -\frac{1100k}{3}$; then

$$R_4 = R - 0.25k\, e^{u_3} \left(e^{-\frac{1100k}{3}R} - 1\right) \quad .$$

In the same fashion as before it can be shown that $0 \leq R_4 <$ R if

$$k^2 \leq \frac{3}{(11)(25)e^{u_3}} \quad . \qquad (116)$$

Thus at each step the condition on k sufficient that $0 \leq R_m < R$ becomes more stringent. Note, however, that it was assumed that $R_2 = R$ in order to obtain (116), and nothing is claimed about (116) being the "best" sufficient condition. Let it be assumed that $R_0 = R_1 = \ldots. = R_m = R$ thence that $E_m = -\frac{(3m+1)100}{3} R$ as would be suggested by the preceding analysis. Then $0 \leq R_{m+1} < R$ if

$$k^2 \leq \frac{3}{(3m+1)25e^{u_m}} .$$

This will certainly be true if

$$k^2 \leq \frac{1}{(m+1)25e^{u_m}} .$$

Since $t_{m+1} = (m+1)k$, this condition can be written

$$k \leq \frac{1}{25t_{m+1}e^{u_m}} . \qquad (117)$$

Let u^* be the minimum of $u(t)$. Then a sufficient condition for R_j to be less than R in magnitude over the whole range: $0 \leq t \leq t_N$ is that

$$k \leq \frac{1}{25t_N\, e^{u^*}} \qquad (118)$$

Entirely similar arguments apply if it is assumed that R is less than zero.

Note also that the total error in u_{m+1} due to errors in $T(x,t)$ is given by

$$E_{m+1} = E_0 - \frac{100k}{3} \; [\; R_0 + 4R_1 + 2R_2 + \ldots + 4R_m + R_{m+1} \;]$$

In fact if $E_0 = 0$ then

$$\left| E_{m+1} \right| \leq (100k)\; mR \quad , \tag{119}$$

where $R = \max\limits_{0 \leq j \leq m+1} \left| R_j \right|$.

Table 1 gives the numerical results obtained by solving (101) and (102) with $T_0 = 0$ and $u_0 = 0.5$. The computation was carried out, using $k = 10^{-3}$ and $k^* = 10^{-5}$. Equation (107) was used for starting the computation of u(t). After u(0.001) had been calculated equation (106) was used for the remainder of the calculation. Originally four terms of (105) were used. Equation (99) was used to check the results and this check is given in column three of Table 1. When only two terms of (105) were used, significant error build-up began to occur and in fact the solutions showed a tendency towards antidamped oscillation. These results and the corresponding results from equation (99) are given in columns four, five and six of Table 1.

A similar calculation was carried out using u_0 = 0.0005 . One would at first think perhaps that no significant information could be gained thereby. However, this case points out a common pitfall of numerical calculations--the "close subtraction." In the computation $e^{u(t)}$ is calculated for use in (105) and then (e^u-1) is computed by subtracting one from e^u. Even though "floating point" calculations are used, the resulting number can have no more than four significant digits if e^u was computed as an eight digit number. No matter how small one makes the truncation errors, no more than a four significant digit result may be obtained. The calculation also shows that errors introduced (by the "close subtractions") do not build-up. The calculations were performed exactly as in the first example. The results are shown in Table 2. Note again that when only two terms of the right-hand side of (105) were retained, significant truncation error build-up occurs. The results are shown to four figures.

Table 1. Solution of the Special Case Equations

with $T_o = 0$ $u_0 = 0.5$

	Four Terms			Two Terms		
t	u(t)	T(t)	$e^u-u+200T^2$	u(t)	T(t)	$e^u-u+200T^2$
0.000	0.50000	0.000000	1.148721	0.50000	0.000000	1.14872
0.100	0.42161	0.015152	1.148721	0.42156	0.015167	1.14878
0.200	0.21552	0.024873	1.148721	0.21517	0.024917	1.14908
0.281	0.00106	0.027269	1.148721	0.00027	0.027331	1.14940
0.400	-0.30618	0.023053	1.148721	-0.30776	0.023119	1.14974
0.500	-0.49755	0.014689	1.148721	-0.49972	0.014739	1.14987
0.636	-0.59980	0.000015	1.148721	-0.60241	0.000026	1.14990
0.800	-0.45312	-0.017315	1.148721	-0.45536	-0.017374	1.14995
0.991	0.00035	-0.027269	1.148721	0.00017	-0.027423	1.15040
1.100	0.28193	-0.022909	1.148721	0.28357	-0.023080	1.15083
1.272	0.50000	-0.000042	1.148721	0.50363	-0.000063	1.15109
1.400	0.37475	0.018553	1.148721	0.37738	0.018727	1.15122
1.554	-0.00097	0.027269	1.148721	-0.00207	0.027548	1.15178
1.800	-0.53446	0.011890	1.148721	-0.54130	0.012038	1.15227
1.908	-0.59980	0.000044	1.148721	-0.60767	0.000084	1.15229
2.100	-0.40168	-0.019729	1.148721	-0.40820	-0.019916	1.15237
2.263	-0.00035	-0.027269	1.148721	-0.00224	-0.027640	1.15279
2.400	0.34209	-0.020362	1.148721	0.34563	-0.020745	1.15332
2.545	0.50000	0.000079	1.148721	0.50730	-0.000003	1.15350

Table 2. Solution of the Special Case Equations

with $T_0 = 0$ $u_0 = 0.0005$

	Four Terms		Three Terms		Two Terms	
t	$u(t)10^3$	$T(t)10^4$	$u(t)10^3$	$T(t)10^4$	$u(t)10^3$	$T(t)10^4$
0.000	<u>0.5000</u>	0.0000	<u>0.5000</u>	0.0000	<u>0.5000</u>	0.0000
0.157	0.3537	0.1767	0.3537	0.1767	0.3536	0.1769
0.314	0.0003	<u>0.2500</u>	0.0003	<u>0.2500</u>	-0.0003	<u>0.2505</u>
0.471	-0.3532	0.1770	-0.3532	0.1770	-0.3547	0.1775
0.628	<u>-0.5001</u>	0.0004	<u>-0.5001</u>	0.0004	<u>-0.5021</u>	0.0004
0.785	-0.3544	-0.1764	-0.3544	-0.1764		
0.943	0.0012	<u>-0.2500</u>	0.0012	<u>-0.2500</u>		
1.100	0.3543	-0.1764	0.3543	-0.1764		
1.257	<u>0.5000</u>	0.0004	<u>0.5000</u>	0.0004		
1.414	0.3530	0.1771	0.3530	0.1771		
1.571	-0.0006	<u>0.2500</u>	-0.0006	<u>0.2500</u>		
1.728	-0.3538	0.1767				
1.885	<u>-0.5001</u>	0.0000				
2.042	-0.3537	-0.1767				
2.199	-0.0004	<u>-0.2500</u>				

A numerical example for the general case.--The following example is considered in this section:

$$
\begin{aligned}
\alpha(x) &= 1 \quad , \qquad |x| \leq 50 \\
\alpha(x) &= 0 \quad , \qquad |x| > 50 \\
n(x) &= 0.01 \quad , \qquad |x| \leq 50 \\
n(x) &= 0 \quad , \qquad |x| > 50 \\
X &= 1/5000 \\
\varepsilon &= 1/25 \\
T_0(x) &\equiv 0 \\
u_0 &= 0.5 \\
k &= 0.0009 \\
k^* &= 0.00009 \\
h &= 0.1
\end{aligned}
\tag{120}
$$

For this example equation (66) takes the form:

$$
\begin{aligned}
T(x,t+k) = &\left(\frac{\lambda^2}{2} - \frac{\lambda}{12}\right) [\, T(x-2h,t) + T(x+2h,t) \,] \\
&+ [\, T(x-h,t) + T(x+h,t) \,] [\, \tfrac{4\lambda}{3} - 2\lambda^2] \\
&+ (1 - \tfrac{5\lambda}{2} + 3\lambda^2)\, T(x,t) + d(x,t) \quad ,
\end{aligned}
\tag{121}
$$

$$d(x,t) = \begin{cases} g(t) = \frac{kn}{\mathcal{E}}\,(e^{u(t)}-1) + \frac{e^{u(t)}k}{2}\,\frac{du(t)}{dt} & |x| \le 50 \\ 0 & |x| > 50 \end{cases} \tag{122}$$

The limits of the integral with respect to x in the power equation are plus and minus fifty. Thus the limit of summation, N, in the x direction which was discussed in Chapter III can be fixed for the entire calculation. Here

$$N = \frac{50}{h} = 500 \quad . \tag{123}$$

The integration of the diffusion type equation was carried twenty increments in h beyond N; that is, it was carried as far as

$$N^{*} = 520 \quad . \tag{124}$$

For the example used this was in effect the "infinite" interval; T(52,t) was identically zero for the number of time steps computed.

The calculations shown for the general case were carried out in fixed-point arithmetic to eight decimal places on the I B M Type 650 calculator. The computations were at a rate of approximately five hundred and fifty mesh points per minute, but because of the smallness of k some twenty-five hours were required to compute the values shown in Table 3 which follows. The seemingly odd choice of k, namely k = 0.0009, was made in order to minimize round-off errors in the calculations involving $\frac{2hk}{9}$, $\frac{hk}{3}$, and similar terms.

In Table 3 are shown sample values for u(t), T(0,t) and T(50,t). The points of relative extreme are underlined. The values shown in this table verify that damping is taking place, and thus that the reactor is dynamically stable. This illustrates the theorem of Ergen and Nohel [16] that the reactor is stable if $\alpha(x)$ and $n(x) > 0$, and ϵ and $X > 0$. In Table 4 is a comparison of u(t) and T(0,t) for $X = \frac{1}{5000}$ with u(t) and T(t) for $X = 0$. Although the comparison is made somewhat difficult by the different values of k used in the two calculations, Table 4 does point up the damping which is taking place in the $X = 0$ case.

Table 3. Numerical Results for the General Case Equations (66) and (72)

t	u(t)	T(0,t)	T(50,t)
0.0000	0.50000	0.000000	0.000000
0.1404	0.35038	0.019937	0.019088
0.2637	0.04818	0.027161	0.024803
0.2817	-0.00084	0.027269	0.024690
0.4230	-0.35734	0.021429	0.017512
0.6363	-0.59972	-0.000016	-0.003835
0.8496	-0.35657	-0.021449	-0.022758
0.9675	-0.06355	-0.027080	-0.026548
0.9909	0.00010	-0.027262	-0.026379
1.1313	0.34921	-0.019989	-0.017484
1.2726	0.49989	0.000057	0.002711

Table 4. A Comparison Table for the

X = 0, X = $^{1}/5000$ Results

t	X = 0 u(t)	X = $^{1}/5000$ u(t)	X = 0 T(t)	X = $^{1}/50000$ T(0,t)
0.0000	0.50000	0.50000	0.000000	0.000000
0.1440	0.34313	0.34313	0.020310	0.020310
0.2810	0.00106	---	0.027269	---
0.2817	---	-0.00084	---	0.27269
0.2820	-0.00167	---	0.027269	---
0.4230	-0.35737	-0.35734	0.021429	0.021429
0.6360	-0.59980	---	0.000015	---
0.6363	---	-0.59972	---	-0.000016
0.6370	-0.59980	---	-0.000098	---
0.8190	-0.41861	-0.41853	-0.018994	-0.018989
0.9900	-0.00237	---	-0.027269	---
0.9909	---	0.00010	---	-0.027262
0.9910	0.00035	---	-0.027269	---
1.1340	0.35463	0.35457	-0.019710	-0.019608
1.2720	0.50000	---	-0.000042	---
1.2726	---	0.49989	---	0.002711
1.2730	0.50000	---	0.000120	---

CHAPTER V

CONCLUSIONS

It has been demonstrated that there is a stable explicit method for solving numerically the system (1), and it has been mentioned that one advantage in this procedure over an implicit one--particularly for digital computers of moderate storage capacity--is that large order systems of linear equations are avoided. However, implicit methods are certainly applicable to the problem and where feasible might well be employed. The investigation of such methods should provide the interested reader a fruitful area of research. In this respect note that implicit methods offer two advantages over explicit methods: one, they have better stability properties, and two, they have higher order truncation errors. At the same time it should be remembered that the stability requirement $\lambda \leq {}^2/_3$ for the method used here is hardly a restriction for the equations considered, since the behavior of the truncation terms dictates that k be small relative to h, and since $\lambda = \frac{X}{\mathcal{E}} \frac{k}{h^2} = 0.005\ {}^k/_{h^2}$.

In order to select an efficient method of solution for problems of the type presented in this study it is of particular importance to examine the partial derivatives of

$T(x,t)$. This is of course difficult and often information can only be obtained from the initial temperature function $T_0(x)$ or by considering a special case. Here for example the derivatives with respect to t are much larger in magnitude than those with respect to x. This would suggest taking $n = 3$ in equation (5) but considering partial derivatives with respect to x of order six--and possibly those of order four as well--to be indentically zero. Then too, a higher order procedure for performing the integration with respect to t in the power equation would be worth while. Using "Weddle's rule," for instance, one obtains

$$u_{m+6} = u_m - \frac{hk}{210}\left[(\Sigma_m + \Sigma_{m+6}) - 6(\Sigma_{m+1} + \Sigma_{m+5}) + 15(\Sigma_{m+2} + \Sigma_{m+4}) - 20\,\Sigma_{m+3} \right] \qquad (125)$$

where

$$\Sigma_{m+j} = (T_{0,m+j} + 4T_{1,m+j} + 2T_{2,m+j} + \cdots + 4T_{N-1,m+j} + T_{N,m+j}). \qquad (126)$$

One would have similar considerations if the derivatives with respect to x were known to be much larger in magnitude than those with respect to t. Thus with modifications suited to the particular problem to be solved, the method presented here should have great practical value for solving problems similar to the system (1).

The results of this study also indicate that numerical procedures are valuable tools in performing mathematical investigations. In particular, by a numerical analysis it was possible to check the property of dynamic stability for a particular reactor for which analytic methods of proving stability could not be justified completely rigorously.

Finally, it should be noted that while in this instance the union of two stable procedures is again stable, this is not necessarily true in general.

BIBLIOGRAPHY

Literature Cited

1. W. K. Ergen, H. J. Lipkin, J. A. Nohel: "Applications of Liapounov's Second Method in Reactor Dynamics," Journal of Mathematics and Physics, XXXVI, No. 1, April 1957, pp. 36-48.

2. W. K. Ergen, and J. A. Nohel: "Stability of a Continuous Medium Reactor," to appear.

3. R. Courant, K. Friedrichs, H. Lewy: "Über die Partiellen Differenzengleichungen der Mathematische Physik," Mathematische Annalen, Vol. 100 (1928), pp. 32-74.

4. Hildebrand, F. B.: Methods of Applied Mathematics, New York: Prentice Hall, Inc., 1952, pp. 319-334.

5. F. John: "Integration of Parabolic Equations," Communications on Pure and Applied Mathematics, Vol. V, No. 2, May 1952, pp. 155-211.

6. Milne-Thomson, L. M.: The Calculus of Finite Differences, Macmillan and Company, Ltd. London, 1933.

7. John, op. cit., p. 160.

8. Ibid., pp. 162-172.

9. Ibid., p. 169.

10. Ibid., p. 162.

11. Courant, Friedrichs, and Lewy, op. cit.

12. John, op. cit., p. 175.

13. Ibid., pp. 186-197.

14. Milne-Thomson, op. cit.

15. Ergen, Lipkin, and Nohel, op. cit., p. 38.

16. Ergen and Nohel, op. cit.

Other References

1. W. G. Bickley: "Difference and Associated Operators, with Some Applications," Journal of Mathematics and Physics, Vol. 27, 1948, pp. 183-192.

2. Gertrude Blanch: "On the Numerical Solution of Parabolic Partial Differential Equations," Journal of Research of the National Bureau of Standards, Vol. 50, No. 6, June 1953, pp. 343-356.

3. Householder, A. S.: Principles of Numerical Analysis, New York, McGraw-Hill, Inc., 1953.

4. Milne, W. E.: Numerical Calculus, Princeton, N. J., Princeton University Press, 1949.

5. Milne, W. E.: Numerical Solution of Differential Equations, New York, John Wiley and Sons, Inc., 1953.

6. G. G. O'Brien, M. A. Hyman, and S. Kaplan: "A Study of the Numerical Solution of Partial Differential Equations," Journal of Mathematics and Physics, Vol. 29, 1951, pp. 223-251.